

# A Sane Proof that $COL_k \leq COL_3$

By William Gasarch

## Abstract

Let  $COL_k$  be the set of all graphs that are  $k$ -colorable. It is well known that  $COL_k$  is NP-complete. It is also well known, and easy, to show that if  $a \leq b$  then  $COL_a \leq COL_b$ . If  $3 \leq a \leq b$  then we also have  $COL_b \leq SAT \leq COL_a$  which is an insane reduction from  $COL_b$  to  $COL_a$ . In this paper we give a sane reduction from  $COL_b$  to  $COL_a$ .

**Keywords:** Graph Coloring, NP-completeness

## 1 Introduction

Let  $A \leq B$  mean  $A$  is polynomial-time reducible to  $B$ .

**Def 1.1** Let  $k \geq 2$ .  $COL_k$  is the set of all graphs that are  $k$ -colorable

Karp [1] showed that  $\{(G, k) : G \in COL_k\}$  is NP-complete. Stockmeyer [3] and Lovasz [2] independently showed that  $COL_3$  is NP-complete.

Assume  $3 \leq a < b$ . It is easy to show that,  $COL_a \leq COL_b$  (add  $K_{b-a}$  and an edge from every vertex of  $K_{b-a}$  to every vertex of  $G$ .) What about  $COL_b \leq COL_a$ ? By the Cook-Levin Theorem  $COL_b \leq SAT$  and since  $COL_a$  is NP-complete  $SAT \leq COL_a$ . Hence  $COL_b \leq COL_a$ . This reduction works but is insane: we transform a graph to a formula and the formula back to a graph. Is there a sane reduction  $COL_b \leq COL_a$ ? There is and we present it here. For all  $k$  we give a sane reduction for  $COL_k \leq COL_3$ .

A proof that does not use formulas is already known. Let  $HCOL_k$  be the set of all hypergraphs that are  $k$ -colorable. Lovasz [2] showed  $COL_k \leq HCOL_2 \leq COL_3$ . Our proof does not use hypergraphs or formulas.

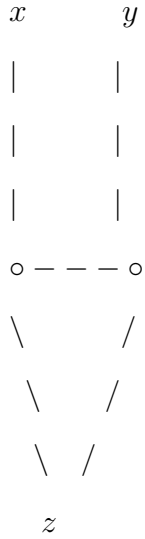
## 2 The Key Gadget

The following gadget is often used to prove that  $COL_k$  is NP-complete.

**Def 2.1**  $GAD(x, y, z)$  is the graph in Figure 1. (The vertices that don't have labels are never referred to so we don't need to label them.)

We leave the proof of the following easy lemma to the reader.

**Lemma 2.2** *If  $GAD(x, y, z)$  is three colored and  $x, y$  get the same color, then  $z$  also gets that color.*



**Figure 1**

**Def 2.3**  $GAD(x_1, \dots, x_k, z)$  consists of  $GAD(x_1, x_2, y_1)$ ,  $GAD(y_1, x_3, y_2)$ ,  $GAD(y_2, x_4, y_3)$ ,  $\dots$ ,  $GAD(y_{k-3}, x_{k-1}, y_{k-2})$ , and  $GAD(y_{k-2}, x_k, z)$ . Aside from  $x_1, \dots, x_k, z$ , the graph  $GAD(x_1, \dots, x_k, z)$  has  $\leq 3k$  vertices, and  $\leq 5k$  edges.

We leave the proof of the following easy lemma to the reader.

**Lemma 2.4** *Let  $k \geq 2$ . If  $GAD(x_1, x_2, \dots, x_k, z)$  is three colored and  $x_1, \dots, x_k$  get the same color, then  $z$  also gets that color.*

### 3 The Main Theorem

**Theorem 3.1** *Let  $k \geq 2$ .  $COL_k \leq COL_3$  by a simple reduction that take a graph  $G$  with  $n$  vertices and  $e$  edges, and produces a graph  $G'$  that has  $\leq 2k^2n + 2ke$  vertices and  $\leq 3k^2n + 2ke$  edges.*

**Proof:** Let  $G$  have vertices  $v_1, \dots, v_n$  and edge set  $E$ . We construct  $G'$ :

1. There are vertices  $T, F, R$  which form a triangle. In any coloring they have different colors which we call  $T, F, R$ . This is 3 vertices and 3 edges. (We won't count these in the end since our crude upper bounds on the vertices and edges in  $G'$  will clearly be over by at least 3.)
2. For  $1 \leq i \leq n$  and  $1 \leq j \leq k$  there is a vertex  $v_{ij}$ . All of these will be connected by an edge to vertex  $R$ . This requires be  $kn$  vertices and  $kn$  edges.
  - (a) For all  $1 \leq i \leq n$  our intent is:  $v_{ij}$  is colored  $T$  means that vertex  $v_i$  in  $G$  is colored  $j$ ;  $v_{ij}$  is colored  $F$  means that vertex  $v_i$  in  $G$  is not colored  $j$ .
  - (b) For all  $1 \leq i \leq n$  we need that *at least one* of  $v_{i1}, \dots, v_{in}$  is colored  $T$ . Hence we need it to not be the case that  $v_{i1}, v_{i2}, \dots, v_{in}$  are all colored  $F$ . We place the gadget  $G(v_{i1}, \dots, v_{in}, T)$  in the graph. If  $v_{i1}, \dots, v_{in}$  are all colored  $F$  then this gadget will not be 3-colorable. This is  $\leq 3kn$  vertices and  $\leq 5kn$  edges.
  - (c) For all  $1 \leq i \leq n$  we need that *at most one* of  $v_{i1}, \dots, v_{ik}$  is colored  $T$ . Hence we need that, for each pair of vertices  $v_{ij_1}, v_{ij_2}$  at most one is colored  $T$ . For each  $1 \leq j_1 < j_2 \leq k$  we place the gadget  $GAD(v_{ij_1}, v_{ij_2}, F)$ . This is  $n \binom{k}{2} \times 2 \leq k^2n$  vertices and  $n \binom{k}{2} \times 5 \leq 2.5k^2n$  edges.
3. For each edge  $(v_i, v_j)$  in the original graph we want to make sure that  $v_i$  and  $v_j$  are not the same color. Place the gadgets  $GAD(v_{i1}, v_{j1}, F), GAD(v_{i2}, v_{j2}, F), \dots, GAD(v_{ik}, v_{jk}, F)$ . This is  $2ke$  vertices and  $5ke$  edges.

Note that the number of vertices in  $G'$  is  $\leq kn + 3kn + k^2n + 2ke \leq 2k^2n + 2ke$  vertices and  $\leq kn + 5kn + 2.5k^2n + 2ke \leq 3k^2n + 2ke$  edges.

Clearly  $G$  is  $k$ -colorable iff  $G'$  is 3-colorable. ■

#### 4 Open Problem

Our reduction takes a graph on  $n$  vertices and  $e$  edges and produces a graph on  $O(n + e)$  vertices and  $O(n + e)$  edges. Can this be improved? For example, is there a reduction that yields a graph with  $O(n + \sqrt{e})$  vertices?  $O(n)$  vertices?

#### 5 Acknowledgment

I would like to thank my students who protested that the reduction  $COL_b \leq SAT \leq COL_a$  was insane and demanded a sane reduction.

#### References

- [1] R. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103, 1972.
- [2] L. Lovasz. Coverings and colorings of hypergraphs. In *Proc. of the 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing*, pages 3–12, 1973.  
[www.cs.elte.hu/~lovasz/scans/covercolor.pdf](http://www.cs.elte.hu/~lovasz/scans/covercolor.pdf).
- [3] L. Stockmeyer. Planar 3-colorability is polynomial complete. *SIGACT News*, 5(1), 1973.